CONVECTIVE INSTABILITIES IN FULLY DEVELOPED FLOWS

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ABSTRACT

This paper considers the possibility of inducing a convective secondary flow in the fully developed channel flow of a quasi-incompressible (Boussinesq) fluid. Instabilities of this type can only occur when the temperature gradient in the direction of the body force exceeds a certain critical value. This temperature gradient is proportional to the Rayleigh number of the fluid. We find that for channels of arbitrary cross section, the critical Rayleigh number is $R_c \geq 1360\ (h/d)^4$ where h is the arbitrary channel's maximum dimension in the body force direction and d is the diameter of an equal area circular channel. For two special geometries it is possible to improve the above lower bound estimate to the critical Rayleigh number. In a circular channel $R_c \geq 3450$ and in a square channel $R_c \geq 2480$.

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INTRODUCTION

Maslen [1] considered the fully developed steady flow of an incompressible fluid in a channel whose generators lie parallel to an axis. He demonstrated that the velocity components normal to the channel axis must vanish everywhere. Velte [2] treated the fully developed steady flow of a quasi-incompressible (Boussinesq) fluid in a similar channel with a horizontal axis. A body force acts along a normal to the axis. The channel wall is nonuniformly heated to establish a constant temperature gradient in the fluid in the direction of the body force ("heated from below"). Velte found that there is a critical value for the Rayleigh number below which the transverse velocity components vanish. The Rayleigh number is proportional to the imposed temperature gradient and is given by $R = g\alpha\beta h^4/k\nu$ where α is the fluid coefficient of thermal expansion, g is the body force per unit mass acting on the fluid, β is the magnitude of the imposed temperature gradient, h is the maximum dimension of the region in the direction of the body force, k is the fluid thermal diffusivity, and

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ν is the fluid kinematic viscosity. Above the critical Rayleigh number it is possible to establish a convective secondary flow.

Velte developed a method for calculating upper bounds to this critical Rayleigh number in channels of arbitrary cross section. If for design purposes it is desirable to insure the stability of the initial fully developed channel flow, then lower bounds to the critical Rayleigh number must be established. In this paper a method is presented to calculate these lower bounds.

THE GOVERNING EQUATIONS

Consider a fully developed quasi-incompressible flow in a channel parallel to the y-axis. The Cartesian velocity components u, v, and w are in the x, y, and z directions (given by the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$). A body force g acts in the negative z direction. The initial velocity, temperature, and pressure distributions in the fluid are

$$\left.\begin{aligned} u = 0, \qquad v &= v_o(x, z)\underset{\sim}{j}, \qquad w = 0 \\ \nabla T &= -\beta\underset{\sim}{k} \\ \nabla p &= -\rho g(1 - \alpha\beta z)\underset{\sim}{k} + \rho\nu\nabla^2 v_o\underset{\sim}{j} \end{aligned}\right\} \quad (1)$$

where T is the temperature, p is the pressure, ρ is the mean fluid density, ∇ is the gradient operator, and ∇^2 is the Laplacian operator.

It is assumed that the secondary convective flow is fully developed (independent of y), hence the governing dimensionless perturbation equations valid at the onset of instability are (see [2])

$$\nabla^2\nabla^2\psi = R\theta_x \quad (2)$$

$$\nabla^2\theta = \psi_x \tag{3}$$

where ψ is the perturbation stream function measured in units of k ($u = -\psi_z$, $w = \psi_x$), and θ is the temperature perturbation in units of βh. Here the gradient and Laplacian operators are in the x, z-plane. If the rigid channel wall has a large thermal conductivity and heat capacity relative to the fluid, then the initial temperature distribution on the wall is maintained at all times. Thus the boundary conditions on the channel wall B are

$$\begin{aligned} \psi = \nabla\psi \cdot \underset{\sim}{n} &= 0 \quad \text{on B} \\ \theta &= 0 \quad \text{on B} \end{aligned} \tag{4}$$

Here $\underset{\sim}{n}$ is a unit outer normal to B. The above eigenvalue problem is equivalent to the variational principle in the cross-section plane C.

$$R = \frac{\int_C (\nabla^2\psi)^2 \, dx\, dz}{\int_C (\nabla\theta)^2 \, dx\, dz}, \qquad \psi = \nabla\psi \cdot \underset{\sim}{n} = \theta = 0 \quad \text{on B} \tag{5}$$

The functions ψ and θ are chosen from Σ_1, a space of admissible functions whose ψ functions are four times differentiable and whose θ functions are twice differentiable. In addition, the functions satisfy the constraint $\nabla^2\theta = \psi_x$ in the region C and the boundary conditions on B. The critical Rayleigh number is given by $R_c = \min R$.

GENERAL LOWER BOUND ESTIMATES TO R_c

Using the inequality $(\nabla\psi)^2 \geq \psi_x^2$, and the constraint $\nabla^2\theta = \psi_x$, it follows directly from equation (5) that

$$R_c \geq \min \frac{\int_c (\nabla^2\psi)^2 \, dx\, dz}{\int_c (\nabla\psi)^2 \, dx\, dz} \cdot \min \frac{\int_c (\nabla^2\theta)^2 \, dx\, dz}{\int_c (\nabla\theta)^2 \, dx\, dz} \tag{6}$$

over the function space Σ_1. Employing the Principle of Monotony [3], one finds that a lower bound to R_c is also given by the extremum problem (6) over a less restrictive function space Σ_2 in which ψ and θ are chosen independent of one another. Hence Σ_1 is a sub-space of Σ_2.

Consider the extremum problem

$$K^2 = \min \frac{\int_c (\nabla^2\psi)^2 \, dx\, dz}{\int_c (\nabla\psi)^2 \, dx\, dz}, \qquad \psi = \nabla\psi \cdot \underset{\sim}{n} = 0 \quad \text{on B} \tag{7}$$

over the function space Σ_2. The smallest extremal value of equation (7) is identical to the principal eigenvalue of the problem posed by

$$\begin{aligned} \nabla^2\nabla^2\psi + K^2\nabla^2\psi &= 0 \qquad \text{in C} \\ \psi = \nabla\psi \cdot \underset{\sim}{n} &= 0 \qquad \text{on B} \end{aligned} \tag{8}$$

For most regions, the principal eigenvalue cannot be calculated in a simple manner. But for a circular channel (in which h = d, the diameter), K^2 is determined from the first root of $J_1(K/2) = 0$ (see [4]) where J_1 is the Bessel function of the first kind of order one. Therefore, $K^2 = 58.74$. Pólya and Szegö [4] have demonstrated that among all plane regions of equal area, the circle has the smallest value of K^2. Thus if one replaces a given arbitrary region of height h by an equal area circle of diameter d, then

$$K^2 \geq 58.74\,(h/d)^2 \tag{9}$$

Next consider the extremum problem

$$L^2 = \min \frac{\int_c (\nabla\theta)^2 \, dx\, dz}{\int_c \theta^2 \, dx\, dz}, \qquad \theta = 0 \qquad \text{on B} \tag{10}$$

over the function space Σ_2. The smallest extremal value of equation (10) is identical to the principal eigenvalue of the following problem:

$$\begin{aligned} \nabla^2\theta + L^2\theta &= 0 \qquad \text{in C} \\ \theta &= 0 \qquad \text{on B} \end{aligned} \tag{11}$$

For a circle, L^2 is determined from the first root of $J_o(L/2) = 0$. Therefore, $L^2 = 23.14$. Pólya and Szegö also have shown that among all plane regions of equal area, the circle has the smallest value of L^2. Analogous to the estimate (9), one then can obtain

$$L^2 \geq 23.14\ (h/d)^2 \tag{12}$$

Using the divergence theorem, the boundary condition $\theta = 0$ on B, and the Schwarz inequality, it follows that

$$\left\{\int_c (\nabla\theta)^2 \, dx\, dz\right\}^2 = \left|\int_c \theta\nabla^2\theta \, dx\, dz\right|^2$$

$$\left|\int_c \theta\nabla^2\theta \, dx\, dz\right|^2 \leq \int_c \theta^2 \, dx\, dz \int_c (\nabla^2\theta)^2 \, dx\, dz$$

Combining these expressions with equation (10) yields

$$L^2 \leq \frac{\int_c (\nabla^2\theta)^2 \, dx \, dz}{\int_c (\nabla\theta)^2 \, dx \, dz} \tag{13}$$

Substituting the estimates (7) and (13) into equation (6) and using the estimates (9) and (12) gives

$$R_c \geq 1360 \, (h/d)^4 \tag{14}$$

As an illustrative example, consider a channel whose cross section is an equilateral triangle of height h. A simple calculation yields $(h/d)^4 = 1.85$ and $R_c \geq 2520$ for this channel.

LOWER BOUNDS TO R_c IN TWO SPECIAL CASES

For channels with circular or square cross sections, the lower bound estimates of the previous section may be improved. In these two special cases we find estimates from equation (6) over the function space Σ_1 rather than the less restrictive function space Σ_2.

The Circular Channel. In a circular channel ($0 \leq r \leq \frac{1}{2}$, $0 \leq \varphi \leq 2\pi$, φ measured counterclockwise from the x-axis) the complete set of eigenfunctions to equation (8) is given by (see ref. [4])

$$\psi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left\{(2r)^m J_m\left(\frac{K_{mn}}{2}\right) - J_m\left(K_{mn} r\right)\right\} \left\{A_{mn} \cos m\varphi + B_{mn} \sin m\varphi\right\} \tag{15}$$

The associated eigenvalues are K_{mn}^2 where K_{mn} is the n^{th} root of $J_{m+1}(K/2) = 0$. Velte [2] has demonstrated that the stream function mode associated with the critical Rayleigh number is symmetric with

respect to both the x- and the z-axis. Thus the subset of (15) that has the proper symmetry property is given by

$$\psi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}\left\{(2r)^m J_m\left(\frac{L_{mn}}{2}\right) - J_m\left(L_{mn}r\right)\right\} \cos m\varphi$$

m an even integer (16)

The principal eigenvalue of the subset is $K^2 = K_{01}^2 = 58.74$.

The complete set of eigenfunctions to equation (11) for the circular channel is given by

$$\theta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(L_{mn}r\right)\left\{A_{mn} \cos m\varphi + B_{mn} \sin m\varphi\right\} \tag{17}$$

The associated eigenvalues are L_{mn}^2 where L_{mn} is the n^{th} root of $J_m(L/2) = 0$. From the constraint $\nabla^2\theta = \psi_x$ and the symmetry property of ψ, one can conclude that θ must be symmetric with respect to the x-axis and anti-symmetric with respect to the z-axis. The complete subset of (17) that has the proper symmetry property is given by

$$\theta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m\left(L_{mn}r\right) \cos m\varphi, \qquad \text{m an odd integer} \tag{18}$$

The principal eigenvalue of the subset is $L^2 = L_{11}^2 = 58.74$. One now has at hand the following estimates for the circular channel

$$K_{01}^2 \le \frac{\int_c (\nabla^2\psi)^2 \, dx\, dz}{\int_c (\nabla\psi)^2 \, dx\, dz}$$

$$L_{11}^2 \le \frac{\int_c (\nabla^2\theta)^2 \, dx\, dz}{\int_c (\nabla\theta)^2 \, dx\, dz}$$

Substituting the above estimates into equation (6) gives

$$R_c \ge 3450 \tag{19}$$

for the circular channel. Recently Sherman [5] used a Rayleigh-Ritz technique and the variational principle (5) to establish the upper bound estimate

$$R_c \le 6510 \tag{20}$$

for the circular channel.

<u>The Square Channel</u>. In a square channel ($|x| \le \frac{1}{2}$, $|z| \le \frac{1}{2}$), the eigenfunctions of equation (8) cannot be determined in any simple form. Weinstein [6] has determined a lower bound to the principal eigenvalue, $K^2 \ge 5.1\, \pi^2$. The associated principal eigenfunction is symmetric with respect to the x- and the z-axis. Velte [2] has shown that the critical stream function mode also has this symmetry.

The complete set of eigenfunctions to equation (11) for the square channel is

$$\theta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{A_m \cos (2m - 1) \pi x + B_m \sin 2m\pi x\right\} \cdot$$

$$\left\{A_n \cos (2n - 1) \pi y + B_n \sin 2n\pi y\right\} \tag{21}$$

As in the circular channel, one can conclude that θ must be symmetric with respect to the x-axis and anti-symmetric with respect to the z-axis. The complete subset of eigenfunctions with the required symmetry is

$$\theta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin 2m\pi x \cos (2n - 1) \pi y \tag{22}$$

The principal eigenvalue of this subset is $L^2 = 5\pi^2$. Thus for the square channel one has the estimates

$$5.1 \pi^2 \leq \frac{\int_c (\nabla^2\psi)^2 \, dx \, dz}{\int_c (\nabla\psi)^2 \, dx \, dz}$$

$$5\pi^2 \leq \frac{\int_c (\nabla^2\theta)^2 \, dx \, dz}{\int_c (\nabla\theta)^2 \, dx \, dz}$$

Substituting the above estimates into equation (6) gives

$$R_c \geq 2480 \tag{23}$$

Velte calculated the upper bound estimate for the square channel

$$R_c \leq 5030 \tag{24}$$

CONCLUDING REMARKS

The lower bounds to R_c established in this paper differ considerably from the upper bounds calculated by the Rayleigh-Ritz method. Nevertheless the lower bounds provide the designer with a limit of error which the Rayleigh-Ritz method cannot yield. Better lower bounds may be established if the Weinstein method [3] can be successfully applied to the variational principle given by equation (5). It is evident from both lower bound and upper bound calculations that the nature of the confining region can have a marked effect on the critical Rayleigh number.

Finally it should be noted that for fully developed flow between two horizontal planes of infinite lateral extent, the critical Rayleigh number is exactly $R_c = 1707.8$ [7].

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